

Trading with the Unborn: A New Perspective on Capital Income Taxation

Kent A. Smetters



Project #: UM03-12

**“Trading with the Unborn:
A New Perspective on Capital Income Taxation”**

Kent A. Smetters
The Wharton School and NBER

March 2004

Michigan Retirement Research Center
University of Michigan
P.O. Box 1248
Ann Arbor, MI 48104

Acknowledgements

This work was supported by a grant from the Social Security Administration through the Michigan Retirement Research Center (Grant # 10-P-98358-5). The opinions and conclusions are solely those of the authors and should not be considered as representing the opinions or policy of the Social Security Administration or any agency of the Federal Government.

Regents of the University of Michigan

David A. Brandon, Ann Arbor; Laurence B. Deitch, Bingham Farms; Olivia P. Maynard, Goodrich; Rebecca McGowan, Ann Arbor; Andrea Fischer Newman, Ann Arbor; Andrew C. Richner, Grosse Pointe Park; S. Martin Taylor, Gross Pointe Farms; Katherine E. White, Ann Arbor; Mary Sue Coleman, ex officio

Trading with the Unborn: A New Perspective on Capital Income Taxation

Kent A. Smetters

Abstract

Security markets between generations are incomplete in the laissez-faire economy since risk sharing agreements cannot be made with the unborn. But suppose that generations *could* trade if, for example, a representative of the unborn negotiated on their behalf today. What would the trades look like? Can government fiscal policy be used to replicate these trades? Would completing this missing market be pareto improving when the introduction of the new security changes the prices of existing assets?

This paper characterizes analytically the hypothetical trades between generations. It shows how the government can replicate these trades by taxing the realized equity premium on investments by either a positive amount or a negative amount. When technology shocks are mostly driven by changes in depreciation, a *positive* tax on the equity premium replicates the hypothetical trades; this tax is also driven by changes in productivity, the choice between a positive and negative tax rate is unclear. However, with log utility, Cobb-Douglas production, and a depreciation rate less than 100 percent, the equity premium is to be taxed at a *negative* rate; this tax is also pareto improving. Finally, simulation analysis is used to consider more complicated cases, including when depreciation and productivity are both uncertain. Under the baseline calibration for the U.S., a *positive* tax on the equity premium is pareto improving.

Authors' Acknowledgements

Helpful comments were received from seminar participants at the 1998 Brown Macroeconomics Workshop; the 1999 IMFseminar series; the 2000 Harvard Monetary and Fiscal Policy Workshop; Centre for Economic Policy Research workshop held in Tilburg, Holland, 2000; 2000 Wharton Macro-Finance workshop, 2000 Wharton Insurance and Risk Management Workshop, as well as John Campbell, Laurence Kotlikoff, Greg Mankiw, and Jan Walliser. This project was supported with a grant from the Social Security Administration through the Michigan Retirement Research Center (UM03-12).

I. Introduction

Security markets in the laissez-faire economy are incomplete due to a “biological” trading constraint that prevents trading with the unborn. As a result, contemporaneous capital market shocks cannot be easily shared with future generations. Symmetrically, future generations cannot share their uncertain future human capital returns with current generations. This market incompleteness might be a source of inefficiency whether these shocks are highly or poorly correlated. This lack of risk sharing might be especially relevant for large shocks, including a large destruction in capital following a terrorist attack or changes in productivity during an economic depression.

But suppose that representatives of future generations traded on their behalf. What would the trades look like? Can the government use its fiscal policy to mimic these trades in order to complete this missing market? Would completing this missing market be pareto improving even though the introduction of this new market may decrease the value of some pre-existing securities?

This paper characterizes the hypothetical trades between generations using a general-equilibrium overlapping-generations model. Shocks arise from two sources – depreciation and productivity – thereby allowing for a large scope of shocks and a range of correlations between wage and capital returns. The lifecycle model herein contains just two periods (work and retirement) in order to focus on inter-generational markets; modeling more than two periods raises several theoretical and computation complications beyond the limits of this paper.¹ Trades between generations are also limited to securities where the payoff is *linear* in the underlying nonlinear stochastic index. This assumption is made in order to derive tractable analytical solutions; it is also well-known that nonlinear contracts can be constructed dynamically from linear contracts in

¹ With two periods, the equilibrium is unique and well defined by the savings behavior of first-period agents under fairly general conditions. With three periods, for example, both first-period and second-period agents can save, creating a complicated fixed point problem. Very few analytical results exist in the literature with more than two periods.

continuous time when the underlying index follows a standard diffusion process. It is unlikely that the key results of this paper would be changed in continuous time.

The paper then shows how the government can replicate these hypothetical trades by taxing the realized equity premium of investments. The realized equity premium is the difference between the investment's actual return and the risk-free return (e.g., a U.S. Treasury bill return) that could have been earned during that period.

When technology shocks are mostly driven by changes in depreciation, we show that the government can replicate the trades that generations would have made by taxing the equity premium at a positive rate. When technology shocks are mostly driven by changes in productivity, the choice between a positive or a negative tax is unclear. But with log utility, Cobb-Douglas production, and a depreciation rate less than 100 percent, the tax rate is negative. In each case, market risk is shared across generations through the government's budget constraint even though the amount of government debt relative to capital is held fixed. In particular, revenue collected from taxing the realized equity premium earned by older workers alters the amount of labor taxes that must be levied on the next generation of younger workers that are alive at the same time.

But would the government's completion of missing markets between generations actually improve efficiency? This question raises two related issues. First, how is efficiency measured? That is, at what "point in time" do we measure the expected utility of those alive today and those born in the future? Expected utility can be measured "ex ante" (before people are born) or at the "interim" (conditional on the state in which people are born). There is considerable disagreement in the literature about the most appropriate approach. Section II attempts to cut through this debate by motivating the "trading approach" to efficiency that stems from simply relaxing the "biological" trading constraint between generations. In effect, this different perspective suggests measuring the

expected utility of those alive at the time of the reform on an *interim* basis while measuring the expected utility of those born in the future on an *ex ante* basis. Of course, in a storage economy, the expected utility of distant future generations is subject to more uncertainty than closer generations.

Second, it is not obvious that completing a missing market improves efficiency. The reason is that the introduction of a new security will change the prices of other securities in the market, potentially creating capital losses for some agents; for example, Hart (1975) gives an example of how the introduction of a new security could actually make everyone worse off. The current paper, however, shows that taxing the equity premium (either positive or negative, depending on the parameters) is typically pareto improving. This result can be proven analytically in some restricted cases of the model. Simulation analysis shows that the results continue to hold more generally.

These results might seem at odd at first with a very large literature on capital income taxes. One strand of the literature argues that investment returns should not be taxed in steady state (Judd, 1985; Chamley, 1986; see the literature review in Auerbach and Hines, 2002). That literature made three key assumptions that are relaxed herein. First, it assumed that households had infinite horizons, thereby ignoring the incompleteness of trading market across generations.² Second, it implicitly focused on taxing the risk-free component of investment returns since there were no *aggregate* risks. Hence, this literature did not comment on the taxation of the risk premium earned by investments. Third, it assumed that households did not face any *idiosyncratic* risks either.

Aiyagari (1995) relaxed this third assumption to allow for idiosyncratic shocks. He showed that precautionary saving increased the steady state capital stock above the modified golden-rule level. A positive tax on capital reduced the capital stock back to its optimal level. A negative tax would never be optimal in his framework. Aiyagari's infinite-horizon model, though, also ignored

² In a deterministic setting, infinite horizons can be rationalized by assuming *altruistic* households. But with aggregate uncertainty, altruism will not generate infinite horizons since *negative* bequests cannot be left in "bad" states.

incomplete markets between generations. It also implicitly focused on taxing the risk-free rate since there were no aggregate risks. The “Aiyagari effect” is not present herein since capital is not over-accumulated.³ Indeed, with only two periods, wages are uncertain only between generations.

Interestingly, a large older literature has already focused on the taxation of the equity premium component of investment returns (Domar-Musgrave [1944]; Tobin [1958]; Mossin [1968]; Stiglitz [1969]; Sandmo [1969, 1977, 1985]; Bulow and Summers [1984]; Gordon [1985]; Kaplow [1994]; Bradford [1995]; Zodrow [1995]; Hubbard and Gentry [1999]); Smetters [2001]; Poterba [2002]; and, Weisbach [2004]). In fact, taxing the equity premium component of investments has traditionally received more attention in the field of public finance since the risk-free rate has historically been near zero; in contrast, the equity premium has averaged five percent or more.⁴

This literature generally concluded that a symmetric tax on the equity premium did not produce efficiency gains or losses since the government shared perfectly in the investment risk. While some of papers in this literature ignored the government’s budget constraint, the papers that accounted for it assumed that the tax revenue was simply rebated lump-sum to the owners of capital. That assumption allowed this literature to focus on a single generation of taxpayers. In contrast, in the model herein, the government raises revenue at each point in time by levying a tax on the capital income earned by second-period agents as well as a tax on the wage income earned by first-period agents. The amount of capital income tax revenue collected directly from one generation, therefore, impacts the amount of wage tax revenue collected from the next generation. In effect, these non-trading generations are, therefore, connected through the government’s budget constraint.

³ In particular, we only consider economies that are interim dynamically efficient.

⁴ Several of these papers argue that the U.S. tax system is approximately Domar-Musgrave in nature due to its generous backward and forward loss combined with the low historic risk-free rate in the U.S. This issue, though, is less important for our purposes since this paper mainly focuses on how the government can complete missing markets and not whether existing tax policy does so. However, this issue will be relevant in the simulation analysis.

An outline of this paper is as follows. Section II illustrates motivates the trading concept of efficiency used herein and explains how it differs from the strict “ex-ante” and “interim” concepts. Section III describes the “Hypothetical Economy” (*HE*) where trading can occur between generations but where the government sector is limited. Section IV outlines the “Replicating Economy” (*RE*) corresponding to the actual market economy where generations cannot trade but where government tax policy can be used to complete this missing market. Section V proves the conditions in the *RE* under which the equity premium is to be taxed at a positive rate and when it is to be taxed at a negative rate. Section VI generalizes the results to a sequential trading economy where generation $t+1$ trades not only with generation t but also with generation $t+2$, which, in turn, trades with generation $t+3$, ad infinitum. Section VII presents some numerical results. Section VIII concludes.

II. The Trading Concept of Efficiency

Figure 1 illustrates the incomplete markets that exists between two adjacent generations in the laissez-faire economy.⁵ Each generation lives two periods; they work during the first period and retire in the second. Generation t agents are alive today at time t and have already received their first-period wage endowment at time t during their first period of life. They then invest some of this income into capital at time t that pays a stochastic rate of return during their second-period retirement at time $t+1$. Generation $t+1$ agents will be born one period from now at time $t+1$, and so they face first-period wages at time $t+1$ that are uncertain as of today at time t . Future national income at time $t+1$, therefore, is divided between the stochastic capital income that will be received by generation t and the stochastic wage income received by generation $t+1$.

Today at time t , generations t and $t+1$ could potentially gain from sharing the risks that

⁵ Related discussions can be found in Muench (1977), Peled (1982), Wright (1987), Gale (1990), Demange and Laroque (1995), and Bohn (1999, 2003).

materialize at time $t+1$.⁶ In particular, generation t faces uncertain capital income returns at time $t+1$ while generation $t+1$ faces uncertain wages. So these two parties might want to enter into some type of risk sharing agreement today in order to share their respective risks at time $t+1$. But a “biological” trading constraint – i.e., generation $t+1$ is not yet born – prevents them. When generation $t+1$ arrives at time $t+1$ there is no more time- $(t+1)$ uncertainty and so the risk sharing opportunity is gone.

But suppose that these generations *could* trade risk-sharing contracts at time t . For example, suppose a court appoints an “executor” that negotiates with generation t on generation $t+1$'s behalf. Generation t would then trade from a position where they know their first-period wage but not their second-period capital returns. Generation $t+1$ (actually its representative today), though, would trade at time t from a position where they do not yet know their first-period wages at time $t+1$.

This thought experiment, therefore, is different from the strict Rawlsian or “ex ante” perspective that is prevalent in the literature. The Rawlsian perspective would require *each* generation to negotiate from behind a “veil of ignorance,” interpreted herein as not knowing their first-period wage endowment. As a result, a pareto improving policy under the Rawlsian approach could actually harm generation t at the wage that it has already earned at time t provided that such a policy was not expected to reduce generation t 's utility before it was born.⁷ For example, a policy that takes from the current rich and gives to the poor is likely pareto improving because the rich could not have expected to have been rich with certainty before they were born. Not surprisingly, therefore, the Rawlsian perspective has drawn considerable criticism in the literature (e.g., Lucas, 1977) because it allows for considerable transfers by forcing agents alive at the time of the reform

⁶ The story is kept simple for now. The discussion is generalized to sequential contracts in Section VI.

⁷ Rangel and Zeckhauser (2001) consider inefficiencies of the private market in a Rawlsian endowment economy. Ball and Mankiw (2001) consider a linear economy. In these economies, the time dimension is unimportant since the level of capital stock is immaterial. In a nonlinear production economy where the amount of capital matters, the Rawlsian experiment requires evaluating the utility of generation t at time $t-1$, immediately before it was born.

to effectively ‘forget their type.’ In contrast, the thought experiment herein simply removes the trading constraint. As a result, a pareto improvement in the hypothetical trading economy herein cannot harm generation t given their wage at time t – even if that wage is quite large. This restriction, therefore, helps to limit the scope for pareto improving risk sharing between generations.

The hypothetical trading economy herein is also different from the strict ‘interim’ perspective that is also often found in the literature. Under the ‘interim’ perspective, a pareto improvement must not decrease the expected utility of generation $t+1$ at each possible wage that they receive at time $t+1$ even if the policy increases their expected utility as of today at time t . For example, a policy that eliminated devastating low wage realizations at time $t+1$ could not qualify as a pareto improvement if it also reduced, even just slightly, wages at time $t+1$ in very good states of the world. As another example, implementing a policy at time t that reduced generation- $(t+1)$ ’s chance of nuclear holocaust could not qualify as pareto improving if generation $t+1$ paid for any of the costs of this policy (e.g., if the policy were deficit financed). The reason is that a nuclear holocaust would not occur in *every* state of the world into which generation $t+1$ was born. Hence, generation $t+1$ would be worse off paying for part of this policy if they *happened* to be born into one of the good states where there was originally (before the policy) sufficiently little or no risk of a holocaust. Although the ‘interim’ perspective does not rule out all risk sharing opportunities between generations, it does severely restrict the allowable set.⁸ In contrast, in the hypothetical trading economy, a pareto gain must simply not reduce the expected utility of generation $t+1$ as of today at time t .

The trading concept of efficiency, therefore, is a mixture of the interim and ex-ante concepts

⁸ As a practical matter, the interim approach is often used because it is more demanding: Policies that increase interim efficiency also increase ex-ante efficiency (Mas-Colell, Whinston and Green, 1995, Section 23.F). However, it is also true that trades that improve *ex-post* efficiency (after all risks are resolved) also increase interim efficiency and, hence, ex-ante efficiency (*Ibid*). The decision to measure utility at the interim position, therefore, seems fairly arbitrary; it appears largely motivated by the fact that the ex-post efficiency is too extreme since it rules out all insurance markets.

of efficiency. The trading concept measures the expected utility of each generation at the same point in time, namely, at time t . In contrast, the ex-ante efficiency concept requires “traveling back in time” for those alive today while the interim concept requires “traveling forward in time” for the unborn.

III. The Hypothetical Economy (*HE*)

This section presents the Hypothetical Economy (*HE*) where generation t can trade a risk sharing contract with the (representative of the) next generation born at time $t+1$.⁹ The economy has three sectors: firms, households and a primitive government with lump sum taxes that issues bonds.

Firms

Net output at time t is produced using capital, K , and labor, L , taking the Cobb-Douglas form:

$$(1) \quad F(K_t, L_t) = A_t K_t^\alpha L_t^{1-\alpha} - \delta_t K_t \quad ,$$

where A is productivity and δ is the depreciation rate. In intensive (per labor unit) form,

$$(2) \quad f(k_t) = A_t k_t^\alpha - \delta_t k_t$$

where $k \equiv K/L$ is the capital-labor ratio. Both A and δ are stochastic, which allows for an imperfect correlation between wage and capital returns (as in Bohn, 1999). $A_t = (1 + a_t) A_{t-1}$ where a_t is a positive bounded i.i.d. random variable, $-1 < \underline{a} < a_t < \bar{a}$, with trend λ . Moreover, $\delta_t = \hat{\delta} + \xi_t$ where $\hat{\delta}$ is a constant and ξ is i.i.d. with a zero mean. Stochastic factor prices are neoclassic,

$$(3) \quad w_t = A_t (1 - \alpha) k_t^\alpha$$

$$(4) \quad e_t = A_t \alpha k_t^{\alpha-1} - \delta_t$$

While some of the analytical results herein don't require assuming this particular specification for technology, at least result does. The simulation analysis also uses this form of technology.

⁹ We generalize to sequential trades between generations $t+1$ and $t+2$, etc., later.

Generation- t Households

Agents live for two periods. In the first-period, generation- t consumers decide how much to save in one-period government debt, s^B , and unleveraged capital, s^K , in order to maximize their time separable and homothetic expected utility over first-period consumption, c_1 , and second-period consumption, c_2 . Conditional on the state of the economy at time t , generation- t 's problem is,

$$(5) \quad \max_{s_t^B, s_t^K, q_{t,t+1}} E_t U(c_{1,t}, c_{2,t+1}) = u(c_{1,t}) + \beta E_t u(c_{2,t+1})$$

$$(6) \quad s.t. \quad c_{1,t} + s_t^K + s_t^B = w_t(1 - \tau_t^W)$$

$$(7) \quad \begin{aligned} c_{2,t+1} &= s_t^K \cdot (1 + e_{t+1}) + s_t^B \cdot (1 + r_{t+1}) \\ &\quad - p_{t,t+1} \cdot q_{t,t+1} + q_{t,t+1} \cdot (e_{t+1} - r_{t+1}) \end{aligned}$$

where the function, $u(c)$, satisfies $\partial u(c) / \partial c > 0$, $\partial^2 u(c) / \partial c^2 < 0$, and $\lim_{c \rightarrow 0} \partial u(c) / \partial c = \infty$. w is the wage rate known today at time t ; τ_t^W is the (lump sum) tax rate on (inelastic) labor income; e_{t+1} and r_{t+1} are the second-period risky return to private capital and the risk-free return, respectively.

$q_{t,t+1}$ is a one-period linear forward contract traded at time t between generation t and generation $t + 1$. This contract pays the realized equity risk premium, $e_{t+1} - r_{t+1}$, at time $t + 1$ and trades at price $p_{t,t+1}$. Payment for the contract is also made at time $t+1$ and both generations overlap. $q_{t,t+1}$ equals the number of contracts purchased by generation t and $q_{t+1,t}$ is the number of units purchased by generation $t+1$. These contracts are obviously in zero net supply: $q_{t,t+1} = -q_{t+1,t}$.

Remark 1 (on the design of the q contract). Three design features of the q contract deserve more discussion. First, the contract's index is specified net of the risk-free rate. In particular, the relevant index is just the realized equity premium, $e_{t+1} - r_{t+1}$, and not the entire return, e_{t+1} . This assumption simplifies the notation without loss in generality: Since the risk-free return is known at time $t+1$, no additional risk-sharing can be achieved by including its value in the index.

Second, the q contract is linear in the underlying index, the realized equity premium. This

assumption allows us to derive tractable analytical solutions; it is also well-known that nonlinear contracts can be constructed dynamically from linear contracts in continuous time when the underlying index follows a standard diffusion process. It is unlikely that the key results of this paper would be changed in a framework with continuous time.

Third, while many securities (e.g., forwards, futures, swaps and options) are single indexed, even more risk sharing would be possible when productivity, A , and depreciation, δ , are *both* stochastic if the q contract's payoff were also indexed to wages at time $t+1$: The number of control indexes would then equal the number of shocks. However, in the analytical results below we consider two boundary cases. In the first case, only capital depreciation is stochastic and so wages are non-stochastic and, hence, are fully predictable. In the second case, only total productivity is stochastic and so capital returns and wages are perfectly correlated. (Section VII reports simulation evidence corresponding to more general cases.) In both of these cases, no improvement in risk-sharing can be achieved by making the payoff of the q contract dependent on wages at time $t+1$. In other words, the q contract fully completes missing markets between generations in these cases. ■

Remark 2 (on bonds). Government debt, s^B , is included in the *HE* for the sake of generality; otherwise, bonds would not exist in a two-period setting with identical first-period agents. Including this debt also allow us to directly compare the *HE* with the Replicating Economy introduced in Section III that will be calibrated to the U.S. economy in Section VII. This safe debt, though, isolates some of the second-period consumption from shocks, and transfers this risk to the next generation; this transfer itself could be inefficient (Bohn, 2003).¹⁰ But, as will be clear, the key analytical results herein do not depend on whether government bonds are in positive or *zero* supply. ■

The first-order conditions for the demand for bonds and equities for given tax parameters are,

¹⁰ Bohn (2003) carefully shows how several existing U.S. fiscal policies aren't efficient under a range of model assumptions. The paper herein then focuses on how new policies can be used to complete missing trading markets.

$$(8) \quad \beta E \left[\frac{u'(c_{2,t+1})}{u'(c_{1,t})} (1 + r_{t+1}) \right] = 1 \quad , \quad \text{and,}$$

$$(9) \quad \beta E \left\{ \frac{u'(c_{2,t+1})}{u'(c_{1,t})} (1 + e_{t+1}) \right\} = 1$$

$$(10) \quad E_t \left[u'(c_{2,t+1}) \cdot (e_{t+1} - r_{t+1}) \right] - E_t \left[u'(c_{2,t+1}) \right] \cdot p_{t,t+1} = 0$$

Representative of Generation-($t+1$) Households at time t

The representative of generation $t+1$ optimizes over each possible state at time $t+1$, $\sigma_{t+1} \in \{A_{t+1}, k_{t+1}, \delta_{t+1}, D_{t+1}\}$, conditional on the state at time t :

$$(11) \quad \begin{aligned} \max_{s_{t+1}^B, s_{t+1}^K, q_{t+1,t}} \quad & E_t \left\{ E_{t+1} \left[U(c_{1,t+1}(\sigma_{t+1}), c_{2,t+2}(\sigma_{t+1})) \right] \right\} \\ & = E_t \left\{ u(c_{1,t+1}(\sigma_{t+1})) + \beta E_{t+1} u(c_{2,t+2}(\sigma_{t+1})) \right\} \end{aligned}$$

s.t.

$$(12) \quad \begin{aligned} c_{1,t+1}(\sigma_{t+1}) + s_{t+1}^K(\sigma_{t+1}) + s_{t+1}^B(\sigma_{t+1}) + p_{t,t+1} \cdot q_{t+1,t} = \\ w_{t+1}(\sigma_{t+1}) \cdot (1 - \tau_{t+1}^W) + q_{t+1,t} \cdot (e_{t+1}(\sigma_{t+1}) - r_{t+1}) \end{aligned}$$

$$(13) \quad c_{2,t+2}(\sigma_{t+1}, \sigma_{t+2}) = s_{t+1}^K(\sigma_{t+1}) \cdot (1 + e_{t+2}) + s_{t+1}^B(\sigma_{t+1}) \cdot (1 + r_{t+2})$$

$w_{t+1} \in \mathcal{W}_{t+1|t}^*$ is unknown at time t but $\inf(\mathcal{W}_{t+1|t}^*) > 0$ since $A_{t+1} > 0$ and so the program (11) - (13) is well defined. (The state index is omitted on time-($t+2$) factor prices and policy functions to reduce notation.) Multiple expectation operators are used (instead of combining them) to emphasize the difference in state space over which generation $t+1$ must optimize relative to generation t .

Generation $t+1$'s first-order conditions are:

$$(14) \quad E_t \left[u'(c_{1,t+1}(\sigma_{t+1})) \right] = \beta \cdot E_t \left\{ E_{t+1} \left[u'(c_{2,t+2}(\sigma_{t+1})) \cdot (1 + e_{t+2} - (e_{t+2} - r_{t+2}) \tau_{t+2}^K) \right] \right\}$$

$$(15) \quad E_t \left[u'(c_{1,t+1}(\sigma_{t+1})) \right] = \beta \cdot E_t \left\{ E_{t+1} \left[u'(c_{2,t+2}(\sigma_{t+1})) \cdot (1 + r_{t+2}) \right] \right\}$$

$$(16) \quad E_t \left[u'(c_{1,t+1}(\sigma_{t+1})) \right] \cdot p_{t,t+1} = E_t \left[u'(c_{1,t+1}(\sigma_{t+1})) \cdot (e_{t+1} - r_{t+1}) \right]$$

Government

Wage (lump sum) tax revenue in the *HE* equals

$$(17) \quad T_{t+1} = \tau_{t+1}^W w_{t+1}$$

where the size of the workforce relative to retirees is stationary; for simplicity, $L_t = 1 \forall t$.

Government debt evolves as

$$(18) \quad D_{t+2} = G_{t+1} - T_{t+1} + (1 + r_{t+1})D_{t+1}$$

where $G_t = G_0 \cdot \left(\frac{f(k_t)}{f(k_0)} \right)$ is government spending. Period $t = 0$ could represent the start of the new policy. Scaling government spending in this way prevents the debt-capital ratio from diverging.

Taxes must be stochastic since, even for a small G_t , the debt-capital ratio can diverge with enough bad shocks. So tax rates are adjusted in order to target a capital-debt ratio, $D_t/K_t \equiv \bar{d}$, for each generation. This policy prevents the government from using budget deficits to share risks across generations; the results herein would be even stronger if this additional degree of freedom were allowed. The wage tax rate at time $t+1$ that stabilizes the debt-capital ratio \bar{d}_{t+2} at \bar{d} equals,

$$(19) \quad \tau_{t+1}^W = \frac{G_{t+1} + (1 + r_{t+1}) \cdot \bar{d} \cdot k_{t+1} - \bar{d} \cdot k_{t+2}}{w_{t+1}}$$

Equation (19) is derived from (18) with $\bar{d}_{t+1} = \bar{d}_{t+2} = \bar{d}$.

Market Clearing

Market clearing requires that the capital stock at time t is equal to the capital saving by private agents. Similarly, government debt must equal the bonds held by the public. Finally, inter-generation contracts must be in zero net supply:

$$(20) \quad k_{t+1} = s_t^k$$

$$(21) \quad D_{t+1} = s_t^B$$

$$(22) \quad q_{t,t+1} = -q_{t+1,t}$$

General Equilibrium

A general equilibrium at time t is composed of a set of household policy rules, $\{c_{1,t}(\cdot), c_{2,t+1}(\cdot), s_t^K(\cdot), s_t^B(\cdot), q_{t,t+1}(\cdot), q_{t+1,t}(\cdot)\}$, the risk-free rate, r_{t+1} , and the equity return distribution, $\Xi(\bar{e}; A_t, \lambda, k_{t+1}, \delta_t, D_t)$, given $\{A_t, k_t, r_t, \delta_t, G_0, \bar{d}\}$, such that:

1. The household's maximization problem, equations (6) - (10), holds.
2. The conditional equity return distribution for e_{t+1} satisfies,

$$\Xi(\bar{e}; A_t, \lambda, k_{t+1}, \delta_t, D_t) = \Pr\left(\{a_{t+1}, \xi_{t+1}\} \in \Theta^a \times \Theta^\xi \mid e(A_{t+1}, k_{t+1}, \delta_{t+1}) < \bar{e}\right)$$

where Θ^a and Θ^ξ are the sets of possible values that the shock terms a and ξ can take. The expression $\Theta^a \times \Theta^\xi$ is the set of all possible states (i.e., the sigma field), and $\Pr(\cdot)$ gives the probability measure of state $\{a_{t+1}, \xi_{t+1}\}$.

3. The wage tax at time t , τ_t^W , satisfies equation (19).
4. The inter-generational market clears: $q_{t,t+1} = -q_{t+1,t}$.
5. The capital market clearing conditions, (20) - (22), hold.

IV. The Replicating Economy (RE)

The Replicating Economy (RE) is the actual economy where trading with the unborn is not possible but where enhanced government policy can be used to replicate these trades.

Households

With capital income taxes, the household problem of generation- t is as follows:

$$(23) \quad \max_{s_t^B, s_t^K, q_{t,t+1}} E_t U(c_{1,t}, c_{2,t+1}) = u(c_{1,t}) + \beta E_t u(c_{2,t+1})$$

$$(24) \quad s.t. \quad c_{1,t} + s_t^K + s_t^B = w_t(1 - \tau_t^W)$$

$$(25) \quad c_{2,t+1} = s_t^K \cdot [1 + e_{t+1} - (e_{t+1} - r_{t+1})\tau_{t+1}^K] + s_t^B \cdot (1 + r_{t+1})$$

τ_{t+1}^K is a Domar-Musgrave (1944) type of capital income tax. It taxes only the realized equity

premium; the tax is also *symmetric* as it taxes the realized premium even if it is negative.

The first-order conditions for the demand for bonds and equities for given tax parameters are,

$$(26) \quad \beta E \left[\frac{u'(c_{2,t+1})}{u'(c_{1,t})} (1 + r_{t+1}) \right] = 1 \quad , \quad \text{and,}$$

$$(27) \quad \beta E \left\{ \frac{u'(c_{2,t+1})}{u'(c_{1,t})} \left[1 + e_{t+1} - (e_{t+1} - r_{t+1}) \tau_{t+1}^K \right] \right\} = 1$$

Combining equations (26) into (27) gives the following equation (derived in the Appendix):

$$(27') \quad \beta E \left[\frac{u'(c_{2,t+1})}{u'(c_{1,t})} (1 + e_{t+1}) \right] = 1$$

The tax on capital income falls out due to symmetry around the risk-free rate.

Government

Total revenue now includes revenue raised from the tax on the equity premium,

$$(28) \quad T_{t+1} = \tau_{t+1}^W w_{t+1} + \tau_{t+1}^K (e_{t+1} - r_{t+1}) s_t^K$$

The wage tax rate at time $t+1$ that stabilizes the debt-capital ratio \tilde{d}_{t+2} at \tilde{d} now equals,

$$(29) \quad \tau_{t+1}^W = \frac{G_{t+1} + (1 + r_{t+1}) \cdot \tilde{d} \cdot k_{t+1} - \tilde{d} \cdot k_{t+2} - \tau_{t+1}^K (e_{t+1} - r_{t+1}) s_t^K}{w_{t+1}}$$

General Equilibrium

The description of general equilibrium is similar to before except for the omission of the inter-generational contracts and the presence of taxes on the equity premium. In particular, a general equilibrium at time t in the *RE* is the set of household policy rules, $\{c_{1,t}, c_{2,t+1}(\cdot), s_t^K, s_t^B\}$, the risk-free rate and equity return distribution for e_{t+1} , $\{r_{t+1}, \Xi(\bar{e}; A_t, \lambda, k_{t+1}, \delta_t, D_t)\}$, given $\{A_t, k_t, r_t, \delta_t, G_0, \tilde{d}\}$, satisfying the following conditions:

1. The household's maximization problem, (24) - (27), holds.

2. The conditional equity return distribution for e_{t+1} satisfies,

$$\Xi(\bar{e}; A_t, \lambda, k_{t+1}, \delta_t, D_t) = \Pr\left(\{a_{t+1}, \xi_{t+1}\} \in \Theta^a \times \Theta^\xi \mid e(A_{t+1}, k_{t+1}, \delta_{t+1}) < \bar{e}\right)$$

where Θ^a and Θ^ξ are the sets of possible values that the shock terms a and ξ can take. The expression $\Theta^a \times \Theta^\xi$ is the set of all possible states (i.e., the sigma field), and $\Pr(\cdot)$ gives the probability measure of state $\{a_{t+1}, \xi_{t+1}\}$.

3. The wage tax at time t , τ_t^W , satisfies equation (29).
 4. The market clearing conditions, (20) and (21), hold.

Existence of a globally unique stochastic stationary equilibrium can be proven á la Wang (1993).¹¹

Replicating the Hypothetical Economy (HE)

We now show how the tax on the equity premium can be chosen in the *RE* to replicate the inter-generational trades in the *HE*. From equations (6) and (7) or (12) and (13) we get:

Proposition 1. Let $s_t^{K^*}$, $q_{t,t+1}^*$, $p_{t,t+1}^*$, r_{t+1}^* , and $\tau_t^{W^*}$ be equilibrium values in the *HE* at time t at a given wage rate w_t . The *RE* economy is then identical to the *HE* economy provided that the tax rates in the *RE* are set as follows: $\tau_{t+1}^{K^*} = -q_{t,t+1}^* / s_t^{K^*}$ and $\tau_t^W = \tau_t^{W^*} + \Delta\tau_t^W$ where $\Delta\tau_t^W = \frac{p_{t,t+1}^* \cdot q_{t,t+1}^*}{(1+r_{t+1}^*) \cdot w_t}$.

Definition 1. $\tau_{t+1}^{K^*} = -q_{t,t+1}^* / s_t^{K^*} = q_{t+1,t}^* / s_t^{K^*}$ is the Replicating Tax (RT) on the equity premium.

In words, a tax on the equity premium can be chosen in the *RE* to share the same risk between generations as the inter-generational q contract in the *HE*. Whereas risk is passed between generations in the *HE* using q contracts, the same risk can be transferred in the *HE* through the government's budget constraint due to the government's ability to tax the equity premium. The change in the wage tax, $\Delta\tau_t^W$, in the *RE* corresponds to the payment for the q contract in the *HE*.

¹¹ Existence of an equilibrium in the *HE* also follows from the equivalence shown in Proposition 1.

V. Signing the Replicating Tax as well as Some Initial Efficiency Results

Since contracts are in zero net supply ($q_{t,t+1} = -q_{t+1,t}$), Proposition 1 shows that the sign of the Replicating Tax (RT) on the equity premium is the same sign as generation $t+1$'s demand for the inter-generational contract, $q_{t+1,t}^*$, at price $p_{t,t+1}^*$. We now use this relationship to sign the RT .

The following two lemmas will be useful. Lemma 1 gives the equilibrium price of the inter-generational contract, $p_{t,t+1}^*$, in the HE that clears the market: $q_{t+1,t}^*(p_{t,t+1}^*) = -q_{t,t+1}^*(p_{t,t+1}^*)$.

Lemma 1. *In the HE , $p_{t,t+1}^* = 0$ and $\Delta\tau_t^W = 0$.*

Proof. Equations (8) and (9) $\Rightarrow E_t[u'(c_{2,t+1}) \cdot (e_{t+1} - r_{t+1})] = 0 \Rightarrow p_{t,t+1}^* = 0$ by (10). The equality $\Delta\tau_t^W = 0$ follows immediately from Proposition 1. ■

Intuitively, the price of the inter-generational contract in the HE must be zero in order for no arbitrage opportunity to exist between generation t 's stock-bond portfolio mix and the inter-generational contract $q_{t,t+1}$. If, for example, $p_{t,t+1}^* > 0$ then generation t could guarantee a profit by selling short the contract ($q_{t,t+1} < 0$) while investing more in stocks and less bonds.

Lemma 2 indicates the sign of the demand of generation $t+1$ for the inter-generational contract in the HE at the equilibrium price.

Lemma 2. *Denote the equilibrium value of $q_{t+1,t}$ at $p_{t,t+1} = 0$, as $q_{t+1,t}^* \Big|_{p_{t,t+1}=0}$. Then, $sign\left(q_{t+1,t}^* \Big|_{p_{t,t+1}=0}\right) = sign\left\{E_t\left[u'(c_{1,t+1}(\sigma_{t+1})) \cdot (e_{t+1} - r_{t+1})\right]\right\}$.*

Proof. Let λ_1 be the Lagrangian multiplier for the constraint $q_{t+1,t} \geq 0$ and let λ_2 be associated with the constraint $q_{t+1,t} \leq 0$. Equation (16) becomes,

$$(30) \quad E_t\left[u'(c_{1,t+1}(\sigma_{t+1})) \cdot (e_{t+1} - r_{t+1})\right] + \lambda_1 - \lambda_2 = 0$$

where $\lambda_1 = 0$ if $\lambda_2 > 0$ and $\lambda_2 = 0$ if $\lambda_1 > 0$. If $\lambda_1 > 0$ then $q_{t+1,t}^* \Big|_{p_{t,t+1}=0} < 0$ and $E_t[\bullet] < 0$. Similarly, if $\lambda_2 > 0$ then $q_{t+1,t}^* \Big|_{p_{t,t+1}=0} > 0$ and $E_t[\bullet] > 0$. It follows that,

$$(31) \quad \text{sign}\left(q_{t+1,t}^* \Big|_{p_{t,t+1}=0}\right) = \text{sign}\left\{E_t\left[u'(c_{1,t+1}(\sigma_{t+1})) \cdot (e_{t+1} - r_{t+1})\right]\right\} \quad . \quad \blacksquare$$

By Definition 1, the right-hand side (RHS) of equation (31) also gives the sign of the Replicating Tax (RT) on the equity premium, $\tau_{t+1}^{K^*}$. In particular, if the RHS of equation (31) is positive, then the sign of the RT that replicates the trades between generations is positive. Conversely, if the RHS of equation (31) is negative, then the RT is negative.

We now derive the sign of the RT analytically for two boundary cases where closed-form solutions are possible. In the first case, depreciation is stochastic and productivity isn't: By equations (3) and (4), wages and capital income, therefore, are *uncorrelated*. In the second case, productivity is stochastic and depreciation is not: Wages and capital income, therefore, are *perfectly* correlated. Both of these variables are allowed to be stochastic in the simulations in Section VII.

Only Depreciation is Stochastic

Proposition 2. *The sign of the RT is positive if depreciation is stochastic ($\xi > 0$) and productivity is not ($\alpha_t = \lambda$) $\forall t$. In other words, the government must tax the realized equity premium at a positive rate in order to replicate trades between generations.*

Proof. To prove this result, we need to show that generation $t+1$ holds a long position in the q -contract at the equilibrium contract price (i.e., $q_{t+1,t}^* \Big|_{p_{t,t+1}=0} > 0$). Decompose $E_t[\bullet]$ in (31) as:

$$(32) \quad E_t\left[u'(c_{1,t+1}(\sigma_{t+1})) \cdot (e_{t+1} - r_{t+1})\right] = E_t\left[u'(c_{1,t+1}(\sigma_{t+1}))\right] \cdot E_t\left[(e_{t+1} - r_{t+1})\right] + \text{cov}\left[u'(c_{1,t+1}(\sigma_{t+1})), (e_{t+1} - r_{t+1})\right]$$

The first ‘‘expected return’’ term on the right-hand side of equation (32) is always positive by non-

satiation ($u' > 0$) and since $E_t[(e_{t+1} - r_{t+1})] > 0$ when equity returns are risky ($\xi > 0$) and agents are risk averse ($u'' < 0$). The second “risk” term is zero since wages are not stochastic in this case and, hence, are uncorrelated with equities. So $sign\{E_t[\bullet]\} > 0 \Rightarrow sign\left(q_{t+1,t}^* \Big|_{p_{t,t+1}=0}\right) > 0$. ■

Intuitively, since expected equity returns faced by generation t at time $t+1$ exceed the risk-free rate, the generation born at time $t+1$ also wants exposure to equity returns at time $t+1$ since those returns are uncorrelated with both their first-period wages at time $t+1$ as well as their second-period equity returns at time $t+2$. A positive tax on risk effectively gives generation $t+1$ access to equity returns at time $t+1$ via the government’s budget constraint: more [less] capital tax revenue collected at time $t+1$ from generation t requires less [more] wage tax revenue from generation $t+1$.

It is generally difficult to analytically prove efficiency gains in general equilibrium even for an endowment economy unless preferences are restricted (see, e.g., Willen, 2002). However, an efficiency gain can be proven herein for stochastic depreciation if we simplify the technology.¹²

Remark 3 (Efficiency Gains with Stochastic Depreciation). The analysis thus far in this section has only assumed that equity returns and wages are uncorrelated; Cobb-Douglas technology is not required. Suppose the technology were linear. Then a positive tax on the equity premium at time $t+1$ is clearly pareto improving. First consider generation t . The tax reduces the volatility of the realized equity premium at time $t+1$, which causes generation t to increase their demand for equities and reduce their demand for bonds by the same amount. At *pre-tax* prices, generation $t+1$ ’s expected utility, therefore, is unchanged since they can achieve the same risk exposure as before.¹³

¹² One reason that we can prove efficiency gains is that the “heterogeneity” herein stems only from having young and old agents alive at the same time, where old agents have already selected their portfolios. In contrast, in Willen’s insightful paper, all two-period agents are young and differ in earnings and preferences. Including that heterogeneity herein would also prevent us from showing efficiency gains; we instead focus on *inter*-generational risk sharing.

¹³ Let $s_t^{K^*}$ and $s_t^{B^*}$ be generation t ’s saving in capital and bonds at the tax rate $\tau_{t+1}^{K^*}$. A new tax rate of $\tau_{t+1}^{K^{**}}$ is announced at time t and let $s_t^{K^{**}}$ and $s_t^{B^{**}}$ equal generation t ’s new desired saving in capital and bonds. At given

However, at *post-tax* prices, the equity return distribution is not changed with linear technology but the risk-free rate rises. So generation t is better off. Now consider generation $t+1$. Their wage rate at time $t+1$ is unchanged; so are other factor prices they face. So generation $t+1$ is better off since equity returns at time $t+1$ are uncorrelated with their wages. Simulation analysis presented in Section VII shows that this results continues to hold in a much more general setting. ■

Only Productivity is Stochastic

This case in which only productivity is stochastic is more intricate. The reason is that the first “expected return” term of (32) is still positive but the second “risk” term is now negative since the marginal utility of consumption is decreasing in wages. So the overall sign is ambiguous.

A closed-form solution can be obtained, though, for the case of log utility, zero net government debt and a deterministic rate of depreciation less than 100 percent (i.e., $\delta < 1$). To simplify notation, the depreciation will be set to zero ($\delta = 0$), although we only strictly need $\delta < 1$.

Proposition 3. *If depreciation is certain and less than full ($\xi = 0$, $\delta < 1$) and productivity is stochastic (i.e., a_t is stochastic), then the sign of the RT is ambiguous. But it is strictly negative for log utility, zero net debt, and Cobb-Douglas production. In other words, the realized equity premium should be taxed at a negative rate in order to replicate inter-generational trades.*

Proof. The ambiguity of the RT in the general case was already explained above. So this proof focuses on the case of log utility, zero net debt, and Cobb-Douglas production. We first derive the expression for the risk-free rate, r_{t+1} . Substitute equation (25) with $s_{1,t}^B = 0$ into (27). With

$$\text{prices, } s_t^{K**} = \left[\frac{(1 - \tau_{t+1}^{K*})}{(1 - \tau_{t+1}^{K**})} \right] \cdot s_t^{K*} \text{ and } s_t^{B**} = s_t^{B*} - (s_t^{K**} - s_t^{K*}) \cdot c_{1,t} \text{ and } c_{2,t+1}(\bullet), \text{ are unchanged.}$$

$u(c) = \ln c$, we get $s_{1,t}^K = \frac{\beta \cdot w_t}{1 + \beta}$ and $c_{1,t} = \frac{w_t}{1 + \beta}$ via equation (24). Combining these expressions with equations (25) and (26) and simplifying,

$$(33) \quad r_{t+1} = \left[E_t \left(\frac{1}{1 + e_{t+1}} \right) \right]^{-1} - 1 \quad .$$

Now substitute equation (33) along with the expressions for the factor returns, (3) and (4), and the policy function $c_{1,t+1}(\sigma_{t+1}) = \frac{w_{t+1}(\sigma_{t+1})}{1 + \beta}$ into the expression $E_t[\bullet]$ shown in equation (31):

$$(34) \quad \begin{aligned} E_t[\bullet] &= E_t \left\{ \frac{1 + \beta}{w_{t+1}} \cdot \left(1 + e_{t+1} - \left[E_t \left(\frac{1}{1 + e_{t+1}} \right) \right]^{-1} \right) \right\} \\ &= E_t \left\{ \frac{1 + \beta}{A_{t+1}(1 - \alpha)k_{t+1}^\alpha} \cdot \left(1 + A_{t+1}\alpha k_{t+1}^{\alpha-1} - \left[E_t \left(\frac{1}{1 + e_{t+1}} \right) \right]^{-1} \right) \right\} \\ &= \left(\frac{1 + \beta}{1 - \alpha} \right) \cdot \frac{\alpha}{s_t^K} \cdot \left\{ E_t \left(\frac{1}{e_{t+1}} \right) \cdot \left[1 - \left[E_t \left(\frac{1}{1 + e_{t+1}} \right) \right]^{-1} \right] + 1 \right\} \end{aligned}$$

Notice that the wage term eventually drops out of equation (34) after some algebraic reduction. This result is due to the perfect correlation between wages and equity returns which allows the sign of $E_t[\bullet]$ to be written as a function of only equity returns at time $t + 1$.

Rearranging the third equality in equation (34) implies,

$$(35) \quad \text{sign}(E_t[\bullet]) = \text{sign} \left\{ E_t \left(\frac{1}{1 + e_{t+1}} \right) \left[1 + E_t \left(\frac{1}{e_{t+1}} \right) \right] - E_t \left(\frac{1}{e_{t+1}} \right) \right\} \quad .$$

We now make use of the following lemma which is proven in the Appendix.

Lemma 3. *Let $x(\omega)$ be a strictly positive real-valued random variable defined on the probability space (Ω, Ψ, P) , $x: \Omega \rightarrow R_{++}$. Then $E\left(\frac{1}{x}\right) \geq E\left(\frac{1}{1+x}\right) \cdot \left[1 + E\left(\frac{1}{x}\right)\right]$, holding with equality if the space is degenerate (Ω contains a single element); holding with strict inequality otherwise.*

Lemma 3 and equations (31) and (35) imply that $q_{t+1,t}^* \Big|_{p_{t,t+1}=0} < 0$ for random equity returns.¹⁴ The RT is, therefore, negative by Definition 1 and so Proposition 3 is proven. ■

Intuitively, a negative tax on the equity premium provides generation $t+1$ with a hedge to their first-period wages. A negative tax raises *more* capital income tax revenue from generation t after a negative productivity shock and *less* revenue after a positive shock. By equation (29), wage taxes on generation $t+1$ are reduced during bad times and increased during good times. Generation $t+1$ pays for improved risk sharing with a larger *expected* wage tax rate since $E_t[(e_{t+1} - r_{t+1})] > 0$.

A non-zero tax rate is somewhat surprising, since the *net* capital return of generation t is already perfectly correlated with generation $(t+1)$'s wage income at time $t+1$. Perfect correlation would, at first blush, seem to rule out any potential for risk sharing.¹⁵ However, for there to be no gain from risk sharing, generation t 's *gross* capital returns at time $t+1$ must be linear in the wage received by generation $t+1$. But that is not the case if the capital saving by generation t has not fully depreciated by the beginning of period $t+1$ (i.e., $\delta < 1$). Only $\delta = 1$ rules out risk sharing gains.¹⁶

Remark 4 (Efficiency Gains with Stochastic Production). In the case of log utility, notice that the policy functions $s_{1,t}^K(w_t) = \frac{\beta \cdot w_t}{1 + \beta}$ and $c_{1,t}(w_t) = \frac{w_t}{1 + \beta}$ do not depend on either the shadow risk-free rate, r_{t+1} , nor on the equity return distribution, $\Xi(\bar{e}; A_t, \lambda, k_{t+1}, \delta_t, D_t)$. The reason is that the income and substitution effects exactly cancel. As a result, a negative tax on the equity premium at time $t+1$ does not impact the capital intensity at time $t+1$, and so does not influence any pre-tax

¹⁴ If equity returns are certain (i.e., $e = r$) then Lemma 1 implies $q_{t+1,t}^* \Big|_{p_{t,t+1}=0} = 0$. Intuitively, there is no value for the Replicating Tax since the tax on risk never raises revenue in this case.

¹⁵ Greg Mankiw helped me think through this particular point. Bohn (1999) is also very informative.

¹⁶ For a depreciation rate of 100 percent ($\delta = 1$) and log utility, $e_t = A_t \alpha k_t^{\alpha-1} - 1 \Rightarrow c_{1,t+1} = \frac{(1-\alpha)}{\alpha \cdot (1+\beta)} \cdot c_{2,t+1}$.

So $E[u'(c_{1,t+1}) \cdot (e_{t+1} - r_{t+1})] = \left[\frac{(1-\alpha)}{\alpha \cdot (1+\beta)} \right] \cdot E[u'(c_{2,t+1}) \cdot (e_{t+1} - r_{t+1})] = 0$, i.e, risk sharing offers no value.

factor prices at time $t+1$. So generation t is unaffected. But a negative tax increases the expected utility of generation $t+1$ if production is Cobb-Douglas and depreciation is not full. Simulation analysis presented in Section VII shows that these results continue to hold in a more general setting. ■

VI. Generalizing to Sequential Trading

The analysis so far has considered risk sharing between two generations: generation t and generation $t+1$. We now show that the Replicating Tax (RT) is still positive [negative] with only depreciation [productivity] shocks if generation $t+1$ also risk shares with generation $t+2$, who is, in turn, risk sharing with generation $t+3$, *ad infinitum*, at time t .

Let $\sigma_t = \{A_t, k_t, \delta_t, D_t, q_{t,t-1}, p_{t-1,t}\}$ denote the state at time t . Set $Z_{s|t}(\sigma_t)$ holds the possible state vectors at time s , conditional on the state at time t . The associated probability measure is $\pi_{s|t}: Z_{s|t}(\sigma_t) \rightarrow [0,1]$. Naturally, the number of vectors contained in $Z_{t|t}(\sigma_t)$, indicated by $\#Z_{t|t}(\sigma_t)$, is one since the time- t state is known at t . If depreciation *or* productivity follows a two-state Markov process, $\#Z_{t+1|t}(\sigma_t) = 2$, $\#Z_{t+2|t}(\sigma_t) = 4$, $\#Z_{t+3|t}(\sigma_t) = 8$, ..., $\#Z_{s|t}(\sigma_t) = 2^{(s-t)}$. If depreciation *and* productivity follow two-state Markov processes then $\#Z_{s|t}(\sigma_t) = 4^{(s-t)}$.

At time t , generation s in the *HE* picks policy functions defined over each state in $Z_{s|t}(\sigma_t)$:

$$(36) \quad \begin{aligned} & \max_{s_s^B, s_s^K, q_{s,s-1}, q_{s,s+1}} E_t \left\{ \left[E_s U(c_{1,s}(\sigma_s), c_{2,s+1}(\sigma_s)) \right] Z_{s|t}(\sigma_t) \right\} \\ & = E_t \left\{ \left[u(c_{1,s}(\sigma_s)) + \beta \cdot E_s u(c_{2,s+1}(\sigma_s)) \right] Z_{s|t}(\sigma_t) \right\} \\ & \text{s.t.} \end{aligned}$$

$$(37) \quad \begin{aligned} & c_{1,s}(\sigma_s) + s_s^K(\sigma_s) + s_s^B(\sigma_s) + p_{s-1,s}(\sigma_{s-1}) \cdot q_{s,s-1}(\sigma_{s-1}) = \\ & w_s(\sigma_s) \cdot (1 - \tau_s^W) + q_{s,s-1}(\sigma_{s-1}) \cdot (e_s(\sigma_s) - r_s) \end{aligned}$$

$$(38) \quad \begin{aligned} & c_{2,s+1}(\sigma_s) + p_{s,s+1}(\sigma_s) \cdot q_{s,s+1}(\sigma_s) = \\ & s_s^K(\sigma_s) \cdot [1 + e_{s+1}(\sigma_{s+1})] + \\ & s_s^B(\sigma_s) \cdot (1 + r_{s+1}(\sigma_s)) + q_{s,s+1}(\sigma_s) \cdot (e_{s+1}(\sigma_{s+1}) - r_{s+1}(\sigma_s)) \end{aligned}$$

The $q_{s,s-1}$ contract and price are indexed to the time-($s-1$) state and $q_{s,s+1}$ is indexed to the time- s state.

An additional constraint is needed to fully specify the dynamic programming problem: the policy function, $c_{2,T+1}(\sigma_{T+1})$, of generation $T \rightarrow \infty$ must be *feasible*. In particular, we must rule out Ponzi games where each generation tries to sell the subsequent generation an overpriced contract relative to the risk sharing provided. Substitute equation (37) into (38) and re-arranging gives:

$$(39) \quad q_{s,s+1} = \frac{\left[q_{s-1,s} \cdot (e_s - r_s - p_{s-1,s}) \cdot (1 + r_{s+1}) - s_s^K \cdot (1 + e_{s+1}) + \left[w_s \cdot (1 - \tau_s^W) - s_s^K - c_{1,s} \right] \cdot (1 + r_{s+1}) + c_{2,s+1} \right]}{(e_{s+1} - r_{s+1} - p_{s,s+1})}$$

where the state index is not shown to reduce notation. Equation (39) is a first-order recurrence relation in q . Integrating (39) forward from date t to T gives the following *realized* value for $q_{T,T+1}$:

$$(40) \quad q_{T,T+1}(\sigma_T) = \left[\prod_{i=t}^{T-1} \frac{\phi_1(i) \cdot (1 + r_{i+1})}{\phi_1(i+1)} \right] \cdot \left[\sum_{i=t}^{T-1} \frac{\phi_2(i+1)}{\phi_1(i+1) \cdot \left[\prod_{j=t}^i \frac{\phi_1(j) \cdot (1 + r_{j+1})}{\phi_1(j+1)} \right]} \right] + q_{t-1,t}(\sigma_{t-1})$$

where

$$\phi_1(i) = e_i(\sigma_i) - r_i(\sigma_i) - p_{i-1,i}(\sigma_{i-1})$$

$$\begin{aligned} \phi_2(i+1) = & -s_i^K(\sigma_i) \cdot (1 + e_{i+1}(\sigma_{i+1})) + \\ & \left[w_i(\sigma_i) \cdot (1 - \tau_i^W(\sigma_i)) - s_i^K(\sigma_i) - c_{1,i}(\sigma_i) \right] \cdot [1 + r_{i+1}(\sigma_i)] + \\ & c_{2,i+1}(\sigma_{i+1}) \end{aligned}$$

The transversality condition is $\lim_{T \rightarrow \infty} \psi(\sigma_T) \cdot p_{T,T+1}(\sigma_T) \cdot q_{T,T+1}(\sigma_T) = 0$ where $\psi(\sigma_T)$ is the Lagrangian multiplier for program (36) - (38), with time index T instead of s . Equation (40) implies:

$$(41) \quad \sum_{i=t}^{\infty} \frac{\phi_2(i+1)}{\phi_1(i+1) \cdot \left[\prod_{j=t}^i \frac{\phi_1(j) \cdot [1 + r_{j+1}(\sigma_j)]}{\phi_1(j+1)} \right]} = 0$$

along with the initial boundary condition, $q_{t-1,t} = 0$, at date t (i.e., today). (41) restricts the space of admissible *policy functions* chosen at time t so that $c_{2,T+1}(\sigma_{T+1})$ is feasible as $T \rightarrow \infty$.

The collective budget constraint in (41) reflects the time- t completeness of the HE with linear securities: An exhaustion of trades between all future generations s and $s + 1$ ($s > t$) at time t implies that trades between generation s and $s + j$ ($j > 1$) at time t are also exhausted.¹⁷ But equation (41) must be interpreted carefully: While agents are “connected” through the inter-generational trading market in the HE , they are not altruistic. For example, a lump-sum transfer between generations without a concomitant change in risk sharing is *not* neutral.

Generation s 's first-order conditions are:

$$(42) \quad E_t \left[u'(c_{1,s}(\sigma_s)) \right] = \beta \cdot E_t \left\{ E_s \left[u'(c_{2,s+1}(\sigma_s)) \cdot (1 + e_{s+1} - (e_{s+1} - r_{s+1}) \tau_{s+1}^K) \right] \right\}$$

$$(43) \quad E_t \left[u'(c_{1,s}(\sigma_s)) \right] = \beta \cdot E_t \left\{ E_s \left[u'(c_{2,s+1}(\sigma_s)) \cdot (1 + r_{s+1}) \right] \right\}$$

$$(44) \quad E_t \left[u'(c_{1,s}(\sigma_s)) \cdot p_{s-1,s}(\sigma_{s-1}) \right] = E_t \left[u'(c_{1,t+1}(\sigma_{t+1})) \cdot (e_{t+1} - r_{t+1}) \right]$$

$$(45) \quad E_t \left\{ E_s \left[u'(c_{2,s+1}(\sigma_s)) \right] \cdot p_{s,s+1}(\sigma_s) \right\} = E_t \left\{ E_s \left[u'(c_{2,s+1}(\sigma_s)) \cdot (e_{s+1} - r_{s+1}) \right] \right\}$$

Equations (42), (43), (44) and (45) are generation s 's first-order conditions for their state-contingent demand for capital (s_s^K), bonds (s_s^B), inter-generational contract with generation $s-1$ ($q_{s,s-1}$), and inter-generational contract with generation $s+1$ ($q_{s,s+1}$), respectively.

Lemma 4. $p_{s,s+1}(\sigma_s) = 0 \quad \forall \sigma_s \in Z_{s|t}(\sigma_t)$ and $\forall s \geq t$.

Proof. Equations (42) and (43) imply $E_t \left\{ E_s \left[u'(c_{2,s+1}(\sigma_s)) \cdot (e_{s+1} - r_{s+1}) \right] \right\} = 0$ and so equation (45) implies $p_{s,s+1}(\sigma_s) = 0 \quad \forall \sigma_s \in Z_{s|t}(\sigma_t)$ and $\forall s \geq t$. ■

Intuitively, similar to before, the zero price for the inter-generational contract rules out

¹⁷ This result is very similar to the equivalence of consumption allocations that exists between the traditional Arrow-Debreu structure in which all contingent claims are traded at date 0 and the sequential (recursive) economy with one-period Arrow securities. See, for example, Ljungqvist and Sargent (2000).

arbitrage opportunities, although now on a state-contingent basis. Also, like before, the sign of the RT at time $s+1$, therefore, is determined by the sign of $q_{s+1,s}$, or $\text{sign}\left\{E_t\left[u'\left(c_{1,s+1}\left(\sigma_{s+1}\right)\right)\cdot\left(e_{s+1}-r_{s+1}\right)\right]\right\}$.

Proposition 4. *If only depreciation is stochastic, the time-($s+1$) Replicating Tax (RT) is positive at every state at time s . In other words, the government should tax the realized equity premium at a positive rate in order to replicate trades between generations. If only productivity is stochastic, the RT is negative in every state at time s for log utility, zero net debt, C-D production, and not full depreciation ($\delta < 1$). In other words, the government should tax the realized equity premium at a negative rate in order to replicate trades between generations.*

Intuitively, the fact that generation $s+1$ now shares risk with generation $s+2$ does not change the *sign* of the risk sharing agreement with generation s under time-separable utility. Specifically, if only depreciation is stochastic, then for each state at time s , generation $s+1$ still wants positive exposure to asset returns faced by generation s since those returns are orthogonal to generation $s+1$'s other risks, including the payoffs to its risk sharing agreement with generation $s+2$. Similarly, with only stochastic productivity, a negative capital income tax still allows generation $s+1$ to hedge its first-period wage income since wages at time $s+1$ are perfectly correlated with capital income at time $s+1$. Since wages at time $s+1$ are uncorrelated with stock returns at time $s+2$ (and, hence, the wages of generation $s+2$), risk sharing with generation $s+2$ does not alter this fact.

VII. Numerical Calculations in the Replicating Economy (RE)

This section calculates efficiency gains numerically in the RE for the more general case in which productivity *and* depreciation are stochastic, allowing for an imperfect correlation between wages and capital returns. In order to calibrate the model to the U.S. economy, the model also includes a social security system where a fraction φ of payroll taxes are deposited into a trust fund

and the other $(1-\phi)$ fraction pays a stochastic wage-indexed pay-as-you-go benefit (the formulae is reported in Smetters, 2001). The pay-as-you-go benefit, like debt, reduces the capital intensity.

Pre-existing capital income taxes are included in order to calibrate the model to the U.S. economy. The pre-existing capital income tax is assumed to also take the Domar-Musgrave form. While this assumption simplifies the analysis, it is also a good approximation of the U.S. tax code (Gordon [1985]; Kaplow [1994]; and Weisbach [2004]). The simulations, therefore, consider changes in this tax rate in order to see if a larger or smaller (maybe negative) tax is optimal.

Each simulation reported below can take up to 20 hours to solve, and so we continue to assume two periods. A many-period production OLG model with *aggregate* uncertainty is difficult to solve due to Bellman's "curse of dimensionality." Two periods allow for exact solutions without resorting to approximations. Any bias in assuming just two periods is unclear. More periods could increase or decrease the magnitude (positive or negative) of the optimal tax rate. On one hand, with more periods a given tax is less effective at inter-generational risk sharing.¹⁸ On the other hand, with more periods, agents can already share risks with younger living agents. Approximation methods with more periods (e.g., Krueger and Kubler, 2001) might provide more insights in the future.

Still, the simulation model advances on previous work in two ways. First, it incorporates an endogenous neoclassical equity return distribution without using approximations; other endogenous variables include capital saving, portfolio choice, risk-free rate, and state-contingent fiscal policies. Second, the model herein reports the welfare change for each generation on the transition path in the presence of aggregate uncertainty where the expected utility of each generation is measured at the reform date. A meaningful metric of the gain or loss is calculated for each generation. Moreover, we can determine whether a policy change is pareto improving even without lump-sum transfers.

¹⁸ The RTR with ∞ horizons is *indeterminate*, not zero (extended Appendix available from author).

Benchmark Calibration

Utility takes the CRRA form, $E_t U_t = \frac{1}{1-\gamma} [c_{1,t}^{1-\gamma} + \beta E_t (c_{2,t+1}^{1-\gamma})]$, where γ is the level of risk aversion and $\beta = 1/(1+\rho)$ where ρ is time preference. Productivity is a two-state Markov process, $A_t = A_{t-1} \cdot (1 + \lambda) \cdot (1 + \tilde{a}_t)$, where λ is trend growth and \tilde{a}_t is a mean-zero stochastic shock, $\tilde{a}_t \in \{\chi, -\chi\}$, that can take values χ and $(-\chi)$ with equal probability. Depreciation is stochastic, $\delta = \hat{\delta} + \varepsilon$, with $\varepsilon \in \{\xi, -\xi\}$. The initial economy at time 0 (the reform date) is summarized in Table 1 and explained in the Appendix. Each period represents 30 years. The correlation between pre-tax wage and capital returns at this low frequency is $3/4$ under the benchmark, which requires both depreciation and productivity to be stochastic. The effective initial tax rate on capital income is set at 20 percent (Auerbach, 1996). The calibration vector, $\{k_0, A_0, \delta_0, \lambda, \gamma, \beta, \chi, \xi, \varphi\}$, generating this initial economy is discussed in the Appendix. In the sensitivity analysis shown below, this vector is re-calculated to generate the same initial economy shown in Table 1.

The Effect of Tax Changes on Macroeconomic Variables

Before turning to welfare calculations, we first consider the macro effects of taxing the equity premium in order to understand how this tax influence prices. Table 2 reports simulation results of the economy along the “mean growth path” where state variables are updated across generations conditional on productivity and depreciation shocks taking their expected values *ex-post*. The table reports the results for two large experiments: reducing the capital income tax from 20 percent to zero as well as doubling its value from 20 percent to 40 percent. Generation 0 are first-period workers alive at the reform, generation 1 are their children born at time 1, etc.

Notice that eliminating capital income taxes reduces the long-run capital stock by 11 percent and output by $3\frac{1}{2}$ percent. The risk-free rate decreases by 20 basis points, the expected return to equities increases by 30 basis points and so the equity premium increases by 50 basis points. These

macro changes have two complementary sources. First, a smaller capital income tax reduces risk sharing across generations and increases the volatility of the realized equity premium faced by first-period agents. Hence, these agents shift their portfolio toward bonds and away from capital, thereby reducing interest rates, increasing equity returns and reducing output. Second, along the mean path, less capital income tax revenue is collected from generation 0, which requires larger wage tax rates to be levied on future generations; this negative wealth effect reduces long-run capital saving.

Notice that, in contrast, doubling capital income taxes increases the long-run capital stock by 25½ percent and output by 6½ percent. The risk-free rate increases by 180 basis points, the expected return to equities decreases by 60 basis points and the equity premium decreases by 240 basis points. However, the speed of convergence is a little slower for this policy change.

Calculating Welfare Gains

While much of the real business cycle literature focuses on mean growth paths like in Table 2 herein, that approach does not capture risk. Along the mean path, generation $t+1$ is unexposed to fiscal policy risk created by generation t . The welfare measure developed in this subsection incorporates that risk. Let a single asterisk (*) denote values before a policy reform. Variables with two asterisks (**) denote values after a policy reform. The *pre-reform* indirect utility of generation s , as calculated at the time of the reform ($t = 0$), is,

$$(46) \quad V_{s|t}^*(\sigma_{t=0}) \equiv \max_{s_s^B, s_s^K, q_{s,s-1}, q_{s,s+1}} E_t \left\{ \left[E_s U(c_{1,s}(\sigma_s, w_s(\sigma_s)), c_{2,s+1}(\sigma_s)) \right] Z_{s|t}^*(\sigma_t) \right\}$$

where $w(\cdot)$ is defined by equation (3). Similarly, the *post-reform* indirect utility at $t = 0$ is

$$(47) \quad V_{s|t}^{**}(\sigma_{t=0}) \equiv \max_{s_s^B, s_s^K, q_{s,s-1}, q_{s,s+1}} E_t \left\{ \left[E_s U(c_{1,s}(\sigma_s, w_s(\sigma_s)), c_{2,s+1}(\sigma_s)) \right] Z_{s|t}^{**}(\sigma_t) \right\}$$

Define the variable $\mu_{s|t}$ as follows:

$$(48) \quad \mu_{s|t} \equiv \mu_{s|t}(\sigma_t) \equiv \left[\frac{V_{s|t}^{**}(\sigma_t)}{V_{s|t}^*(\sigma_t)} \right]^{1-\gamma}$$

The following equality can be shown to hold:

$$(49) \quad V_{s|t}^{**}(\sigma_{t=0}) \equiv \max_{s_s^B, s_s^K, q_{s,s-1}, q_{s,s+1}} E_t \left\{ \left[E_s U(c_{1,s}(\sigma_s, \mu_{s|t} \cdot w_s(\sigma_s)), c_{2,s+1}(\sigma_s)) \right] Z_{s|t}^*(\sigma_t) \right\}$$

Notice that V has two asterisks, indicating post-reform expected utility, while the Z term has a single asterisk, indicating the pre-reform state space. Also note the multiplier $\mu_{s|t}$ on wages at time s .

In words, the expected utility of generation s after the policy change equals what their expected utility would be with no policy change if, instead, each possible wage at time s , measured at t ($t \leq s$), is multiplied by $\mu_{s|t}$. When $s = t$, the time- s state is known and $\mu_{s|t}$, therefore, gives the usual Equivalent Variation measure used in single-state models (e.g., Auerbach and Kotlikoff, 1987).

Notice that the welfare measure for each generation – including those born on the transition path and in each steady state – is calculated at the time of the reform, time $t = 0$. As a result, generation $s > 0$ born further into the future faces more uncertainty than a generation \tilde{s} born closer to the policy-reform date, where $s > \tilde{s} > 0$. The nonlinear forms for utility and technology require simulating each path that the economy can take between times t and when each generation s is born, with each path weighted by its probability of occurring. In particular, for discrete Markov processes,

$$(50) \quad \begin{aligned} V_{s|t}^*(\sigma_{t=0}) &\equiv \max_{s_s^B, s_s^K, q_{s,s-1}, q_{s,s+1}} E_t \left\{ \left[E_s U(c_{1,s}(\sigma_s), c_{2,s+1}(\sigma_s)) \right] Z_{s|t}^*(\sigma_t) \right\} \\ &= \sum_{\sigma_{t+1} \in Z_{t+1|t}^*(\sigma_t)} \sum_{\sigma_{t+2} \in Z_{t+2|t+1}^*(\sigma_{t+1})} \cdots \sum_{\sigma_s \in Z_{s|t}^*(\sigma_{s-1})} \\ &\quad \left\{ \Pi(\sigma_{t+1}, \sigma_{t+2}, \dots, \sigma_s | \sigma_t) \cdot \max_{s_s^B, s_s^K, q_{s,s-1}, q_{s,s+1}} E_s \left[U(c_{1,s}[\sigma_s, w(\sigma_s)], c_{2,s+1}[\sigma_s]) \right] \right\} \end{aligned}$$

where $\Pi(\{\sigma_i\}_{i=t+1}^{s+1} | \sigma_{t=0})$ is the joint probability of the sequence $\{\sigma_i\}_{i=t+1}^{s+1}$, given the state of the economy at time $t = 0$. (The calculation of V^{**} is similar except with Z^{**} state spaces.) The total

number of paths equal $4^{(s-t)}$ for the two-state discrete processes for productivity and depreciation outlined above. So, for example, calculating $V_{5|0}$, the expected utility for the cohort born five generations (150 years) after time 0, requires simulating 1024 ($= 4^5$) general-equilibrium paths.

The different paths are calculated recursively along a dense lattice structure shown in Figure 2. Each node on the lattice satisfies the conditions for a general equilibrium in the *RE* shown in Section IV (Social Security adds more conditions). The equilibrium state vector σ calculated at each “parent node” is passed to four (two \times two-state Markov processes) “child nodes” where each child represents a possible state the economy can take the next period, conditional on being at that parent. Recursion stops when $|\mu_{s+1|t} - \mu_{s|t}| < \varepsilon$, where ε is small, which occurs after about five generations (150 years) for the simulations herein. The corresponding lattice contains 5,461 nodes.

Simulation Results

Table 3 reports the percent change in welfare, $[(\mu_{s|t=0} - 1) \cdot 100\%]$, for generations $s \in \{0, 1, \dots, 5\}$ corresponding to the two tax reforms considered in Table 2. Recall that generation 0 is a first-period worker alive at the reform time 0, generation 1 is their future children, followed by generation 2, etc. The change in welfare for generation (-1) agents, retired at the time of the reform, is always zero since their portfolio decisions and returns are known at the time of the policy change.

Notice that lower the capital income tax from 20 percent to 0 at $t=1$ reduces the welfare of future generations by 8½ percent. Convergence is fairly quick. Notice, though, that doubling the tax rate to 40 percent also lowers future welfare, now by 2 percent. Convergence, though, is slower. In other words, these opposite experiments both serve to reduce the welfare of future generations.

To understand these results, recall that a positive capital income tax gives future generations exposure to previous stock returns through the government’s budget constraint: Future *expected*

after-tax wages increase as do their *volatilities*. Evidentially, even at a $\frac{3}{4}$ correlation between pre-tax wages and capital income, wages and capital income are still not correlated enough to warrant a smaller tax rate. The reason is that the welfare of future generations is influenced more by the expected increase in after-tax wages than by the increased volatility. But, at a 40 percent tax rate, future generations are exposed enough to past stock returns and so volatility becomes more important. In sum, future generations don't want a tax rate that is either too low or too high.

Table 3 also shows that increasing the capital income tax rate to only 30 percent, however, gets this tradeoff just right. Not only are future generations better off, so is the initial generation of young agents at time 0. So this policy change is a pareto improvement. Generation 0 gains from the higher risk-free rate as they shift their portfolio from bonds to stocks in response to the greater risk sharing provided by the larger tax. Generations s , for $s > 1$, gain from additional exposure to previous stock returns since those returns are not correlated enough with their first-period wages even at a $\frac{3}{4}$ correlation. If the baseline correlation were reduced below $\frac{3}{4}$, then depreciation shocks play a more important role in the economic uncertainty. In this case, an even larger tax increase is pareto improving (not shown for brevity). These results suggest that not only is the Replicating Tax on the equity premium positive when depreciation shocks become more important (see Proposition II), the optimal tax rate is also very large in magnitude under the baseline calibration.

More Sensitivity Analysis: A Larger Correlation Between Wages and Capital Returns

When the model is re-calibrated so that the correlation between wage and capital returns along the mean growth path is increased from $\frac{3}{4}$ to 0.80 (not shown), the tax rate that maximizes long-run welfare is closer to the 20 percent tax rate used under our baseline rather than 30 percent. This result is robust to many model and parameter changes including: (i) government spending in

the utility function;¹⁹ (ii) a reasonable range of values for the risk-free rate and equity premium along the mean growth path; and (iii) starting the economy from a zero tax rate and increasing it to 20 and then 40 percent.²⁰ Interestingly, 0.80 is almost exactly the point estimate between wage and stock returns for 30-year moving averages since 1929, and so the current effective tax rate in the U.S. appears to maximize long-run expected utility at reasonable parameter values. The 0.80 correlation estimate, though, is based on two and half unique 30-year periods, and so the standard errors are large. In particular, a correlation of unity cannot be rejected.

Table 4 reports sensitivity analyses in which the correlation between wages and capital income along the mean path is set at unity (i.e., only productivity shocks are operative). Eliminating the tax on the equity premium is now *nearly* pareto improving. The welfare gain to generation 0 is negative since the risk-free rate increased slightly (recall that we dropped the assumption of log utility). But the change in their welfare is so small that it is rounded in Table 4 to zero (i.e., < -0.05 percent). The welfare gains to all future workers, though, are large. So, with only a very small lump-sum transfer from future workers to current workers, the policy would be pareto improving.

Table 4 also shows that setting the tax rate equal to negative 20 percent improves the welfare of all future generations by even more (1½ percent), with very little impact on current workers (-0.1 percent). Long-run welfare is, in fact, maximized at about - 20 percent. This result shows that relaxing the assumptions made in the earlier theoretical analysis (log utility, zero net bonds, etc.) does little to undermine the case for a negative tax rate when only productivity shocks are operative.

¹⁹ Utility with government consumption, G , is $\left[(c_{1,t})^{1-\gamma} + \beta_G (G_t)^{1-\gamma} + \beta E_t [c_{2,t+1}]^{1-\gamma} \right] / (1-\gamma)$ where β_G is set so that the marginal utility of government spending equals the marginal utility of first-period consumption (and, hence, the expected marginal utility of second-period consumption) calibrated for generation-0 agents: $\beta_G \equiv (G_0^* / c_{1,0}^*)^\gamma$.

²⁰ The model is always re-calibrated to hit the other aforementioned targets.

VIII. Conclusions

This paper started with a thought experiment: suppose that living generations could trade risk sharing contracts with the next unborn generation. What would these trades look like? The paper then demonstrated that the government can replicate these trades using a tax on the equity premium. Conditions were derived under which the replicating tax is positive and when it is negative.

However, in general, completing a missing market is not necessarily pareto improving since existing assets can lose some of their value. Nonetheless, it was shown a tax on the equity premium is generally pareto improving in the most strict sense: All generations are better off. In some simplified cases, analytical solutions were possible. For more general cases, simulation analysis was used. Under our benchmark calibration for the U.S., the efficient capital income tax rate is positive and large. Indeed, a large tax on the equity premium is optimal even at a high correlation between wages and stock returns (exceeding 0.80). Only at very high correlations is the optimal tax negative.

IX. Appendix

Derivation of Equation (27')

Equations (27) and (26) imply:

$$\beta E \left\{ \frac{u'(c_{2,t+1})}{u'(c_{1,t})} \left[1 + e_{t+1} - (e_{t+1} - r_{t+1}) \tau_{t+1}^K \right] \right\} = 1 = \beta E \left[\frac{u'(c_{2,t+1})}{u'(c_{1,t})} (1 + r_{t+1}) \right] \text{ by (26)}$$

$$\Rightarrow \beta E \left\{ \frac{u'(c_{2,t+1})}{u'(c_{1,t})} \left[1 + e_{t+1} - (1 + r_{t+1}) - (e_{t+1} - r_{t+1}) \tau_{t+1}^K \right] \right\} = 0$$

$$\Rightarrow \beta E \left\{ \frac{u'(c_{2,t+1})}{u'(c_{1,t})} \left[(1 - \tau_{t+1}^K) \cdot (e_{t+1} - r_{t+1}) \right] \right\} = 0$$

$$\Rightarrow \beta E \left\{ \frac{u'(c_{2,t+1})}{u'(c_{1,t})} \left[(1 + e_{t+1}) - (1 + r_{t+1}) \right] \right\} = 0$$

$$\Rightarrow \beta E \left\{ \frac{u'(c_{2,t+1})}{u'(c_{1,t})} (1 + e_{t+1}) \right\} - 1 = 0 \quad \text{by (26).}$$

Proof of Lemma 3²¹

The inequality shown in Lemma 3 clearly holds with equality if x is non-random. If x is random then let $\varsigma \equiv (1/x)$ define the function $g(\varsigma) \equiv \varsigma/(1+\varsigma)$. The expression becomes $E(\varsigma) \geq E\left(\frac{\varsigma}{1+\varsigma}\right) \cdot [1 + E(\varsigma)]$, or, $\frac{E(\varsigma)}{[1 + E(\varsigma)]} \geq E\left(\frac{\varsigma}{1+\varsigma}\right)$, which holds with strict inequality by Jensen's inequality since g is concave. ■

Calibration of the Benchmark Economy (Table 1)

The expected annual depreciation equals 5 percent so that 79 percent of the capital stock is *expected* to be depreciated by the end of a 30-year period. The capital share, α , is set at 0.30. The

²¹ Thanks to Greg Nini for this succinct proof.

arbitrary scaling parameter A_0 equals unity. Based on Poterba (1998) and Ibbotson data, the annual pre-tax (social) real rate of return to capital equals 8½ percent per year, or 1,056 percent over 30 years, with a coefficient of variation equal to 0.87. The annual risk-free real return, r_1 , equals 3 percent, or 143 percent over 30 years, based on historic returns to long-term government securities this century.²² The annual expected rate of labor-augmenting technological progress is set at 3 percent per year, the average growth rate of the total salaries and wage base since 1929, based on Bureau of Economic Analysis data. The point-estimate correlation between wage and stock returns at a 30-year frequency is about three-quarters. The debt-capital ratio, \bar{d} , is set at 0.25, close to the current ratio of government debt relative to the domestically-owned capital stock as measured in the Federal Reserve Board’s Flow of Funds Accounts. The initial tax rate on the generation-0 agent’s second-period capital income, τ_1^K , is set at 0.20, following the careful calculations found in Auerbach (1996). The initial proportional tax rate on wage income, τ_0^W , is set at 0.15 which generates a plausible level of tax revenue derived from wages. The Social Security payroll tax is set at 12 percent and the estimated ratio of contributions to the Social Security trust fund divided by benefits paid during the past and next few decades equals about 4 percent.

Calibrating the model involves “inverting” many of the equations presented in Section III to express the parameter vector $\{k_0, A_0, \delta_0, \lambda, \gamma, \beta, \chi, \xi, \varphi\}$ as a function of the economic variables to be targeted at time 0. The resulting parameter vector is unique. The calibrating vector needed to generate the baseline economy described in the previous paragraph is $\{k_0, A_0, \delta_0, \lambda, \gamma, \beta, \chi, \xi, \varphi\} = \{.0056, 1.0, 0.79, 0.860, 0.857, 0.27, 0.61, 6.07, 0.04\}$. The value $\beta = 0.27$ corresponds to an *annual* rate of time preference equal to 4.4 percent. The value of $\gamma = 0.857$ reflects several factors: scaling

²² The exact choice of the risk-free rate is not so important if the model is properly recalibrated each time. A relatively higher return corresponding to long-term debt, versus short-term debt, is used under our benchmark to avoid potential criticism in the sensitivity analysis where the Domar-Musgrave assumptions are relaxed.

the model to equity returns (rather than consumption data); human capital depreciation in the second period; and a three-quarters correlation of wage-indexed pay-as-you-go Social Security returns with stock returns. The simultaneous equation set outlined in Section III is solved using a Jacobian-based generalized Newton method. Additional details are available from the author.

The model calibration generates additional plausible economic relationships (Table 1). The implied net national saving rate equals a realistic 4.4 percent. The non-Social Security part of government spending equals 15.3 percent which is very close to the value of 15½ percent that the CBO (1999) reports for 1998. Capital income tax revenue equals 4.4 percent of GDP while wage income taxes, not including Social Security payroll taxes, compose 10½ percent of GDP.

References

- Aiyagari, S. Rao. "Optimal Capital Income Taxation with Incomplete Markets, Borrowing Constraints, and Constant Discounting." *Journal of Political Economy*, 1995, 106: 1158-1175.
- Auerbach, Alan, "Tax Reform, Capital Accumulation, Efficiency, and Growth," in Henry J. Aaron and William G. Gale, eds., Economic Effects of Fundamental Tax Reform, The Brookings Institution: Washington, D.C., 1996, 29-81.
- Auerbach, Alan and Laurence Kotlikoff. Dynamic Fiscal Policy, Cambridge: Cambridge University Press, 1987.
- Auerbach, Alan and James Hines. "Taxation and Economic Efficiency," in Alan Auerbach and Martin Feldstein, Editors, Handbook of Public Economics, 2002, Vol 3, 1347- 1421.
- Ball, Laurence and N. Gregory Mankiw. "Intergenerational Risk Sharing in the Spirit of Arrow, Debreu, and Rawls, with Applications to Social Security Design." NBER WP #8270, 2001.
- Bohn, Henning. "Should the Social Security Trust Fund hold Equities? An Intergenerational Welfare Analysis." *Review of Economic Dynamics*, vol.2, no.3, July 1999, 666-697.
- Bohn, Henning. "Intergenerational Risk Sharing and Fiscal Policy." UCSB Mimeo, 2003.
- Bradford, David. "Consumption Taxes: Some Fundamental Transition Issues." NBER Working Paper No. 5290, 1995.
- Bulow, Jeremy and Lawrence Summers. "The Taxation of Risky Assets." *Journal of Political Economy*, 92, 20, 1984.
- Congressional Budget Office. "The Economic and Budget Outlook: An Update" July 1, 1999.
- Demange, Gabrielle and Guy Laroque. "Optimality of Incomplete Markets." *Journal of Economic Theory*, 65, 1995: 218 - 232.
- Domar, E. and R. Musgrave. "Proportional Income Taxation and Risk Sharing." *Quarterly Journal of Economics*, 1944, 58, 388 - 422.
- Gale, Douglas. "The Efficient Design on Public Debt." in R. Dornbusch and M. Draghi, Public Debt Management: Theory and History, Cambridge University Press, 1990.
- Gordon, Roger. "Taxation of Corporate Capital Income: Tax Revenues versus Tax Distortions." *Quarterly Journal of Economics*, 100, 1985, 1 - 27.
- Hart, Oliver. "On the Optimality of Equilibrium when the Market Structure is Incomplete." *Journal of Economic Theory*, 1975, 11: 418-435.
- Hubbard, R. Glenn and William Gentry. "Fundamental Tax Reform and Corporate Financial Policy,"

in James M. Poterba, ed., *Tax Policy and the Economy*, Volume 12.

Kaplow, Louis. "Taxation and Risk-Taking: A General Equilibrium Perspective." *National Tax Journal* 47, 135, 1994.

Krueger, Dirk and Felix Kubler. "Intergenerational Risk Sharing via Social Security when Financial Markets are Incomplete." Mimeo, Stanford University.

Ljungqvist, Lars and Thomas J. Sargent. Recursive Macroeconomic Theory, MIT Press, 2000.

Lucas, Robert E., Jr. "Reply to Muench and Polemarchakis and Weiss (1977)." *Journal of Economic Theory*, 1977, 15: 351 - 352.

Mas-Colell, Andreu, Michael D. Whinston, and Jerry R. Green. Microeconomic Theory, Oxford University Press: New York, 1995.

Muench, Thomas J.. "Efficiency in a Monetary Economy." *Journal of Economic Theory*, 1977, 15: 325 - 344.

Mossin, J. "Taxation and Risk-Taking: An Expected Utility Approach," *Economica*, 1968, 35, 74 - 82.

Peled, Dan. "Informational Diversity Over Time and the Optimality of Monetary Equilibria." *Journal of Economic Theory*, 1982, 28: 255 - 274.

Poterba, James. "The Rate of Return to Corporate Capital and Factor Shares: New Estimates Using Revised National Income Accounts and Capital Stock Data." NBER, #6263, 1998.

Poterba, James. "Taxation, Risk-Taking, and Household Portfolio Behavior," In Alan Auerbach and Martin Feldstein, Eds., Handbook of Public Economics, 2002.

Rangel, Antonio and Richard Zeckhauser. "Can Market and Voting Institutions Generate Optimal Intergenerational Risk Sharing?" in J. Campbell and M. Feldstein, eds., *Risk Aspects of Investment Based Social Security Reform*, Chicago University Press, 2001.

Sandmo, Agnar. "Capital Risk, Consumption, and Portfolio Choice," *Econometrica*, 1969, 37, 586-599.

Sandmo, Agnar. "Portfolio Theory, Asset Demand and Taxation: Comparative Statics with Many Assets," *Review of Economic Studies*, 1977, 44, 369 - 379.

Sandmo, Agnar. "The Effects of Taxation on Savings and Risk Taking." in A.J. Auerbach and M. Feldstein, eds., Handbook of Public Economics, Vol I., 1985, 265 - 311.

Smetters, Kent. "The Equivalence of the Social Security Trust Fund's Portfolio Allocation and Capital Income Tax Policy." NBER Working Paper #8259, 2001.

Stiglitz, Joseph. "The Effects of Income, Wealth and Capital Gains Taxation on Risk-Taking." *Quarterly Journal of Economics*, 1969, 83, 262 - 283.

Tobin, James. "Liquidity Preference as Behavior Towards Risk." *Review of Economic Studies* 25, 65, 1958.

Wang, Y. "Stationary Equilibria in an Overlapping Generations Economy with Stochastic Production." *Journal of Economic Theory*, 61, 1993, 423 - 435.

Wright, Randall. "Market Structure and Competitive Equilibrium in Dynamic Economics Models," *Journal of Economic Theory*, 1987, 41: 189 - 201.

Weisbach, David. "Taxation and Risk-Taking with Multiple Tax Rates." *National Tax Journal*, 2004.

Willen, Paul. "New Financial Markets: Who Gains and Who Loses." Mimeo, 2002. Forthcoming in *Economic Theory*.

Zodrow, George. "Taxation, Uncertainty and the Choice of a Consumption Tax Base." *Journal of Public Economics*, 1995, 58: 257 - 265.

Figure 1
The Division of National Income in an Overlapping-Generations Economy

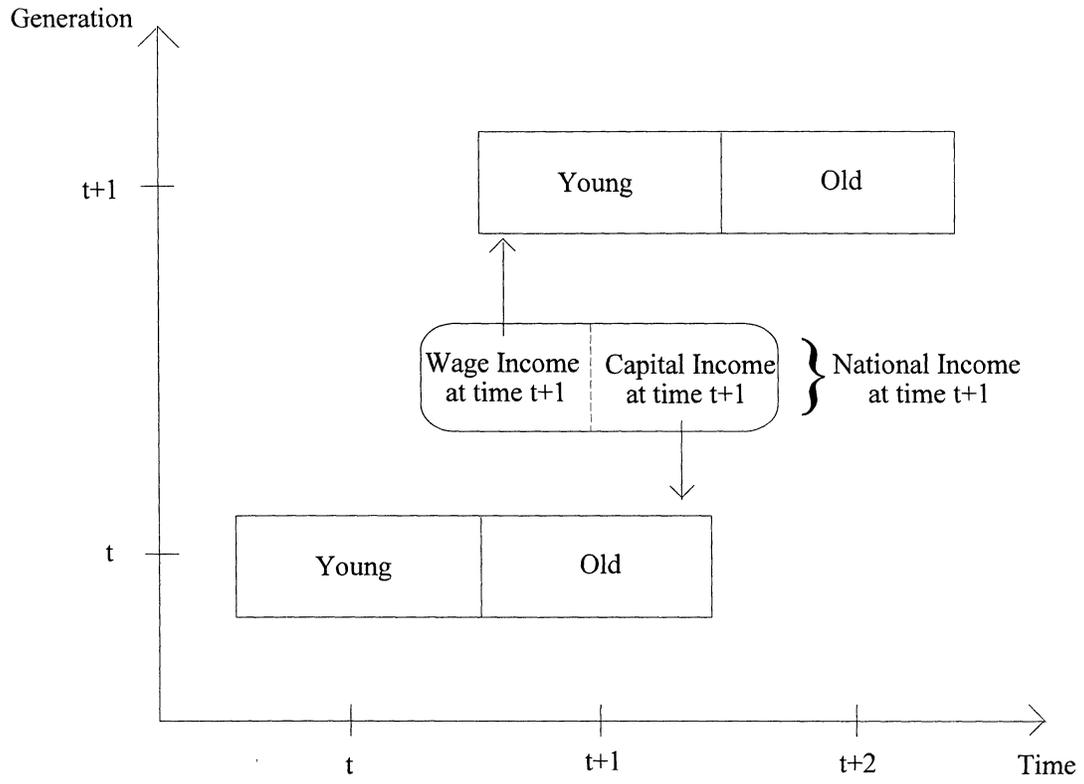


Table 1
Parameters and Implied Values

Variable Description	Value
Exogenous Parameters (same in all simulations, unless indicated otherwise)	
Average annual depreciation rate, $\bar{\delta}_{annual}$	5 %
Capital share, α	0.30
Arbitrary Scaling of the Initial Productivity, A_0	1.00
Pre-tax 30-year return to equities on mean path, $\overline{E(e_1 k_1)}$ (Corresponding annual return)	1,056 % (8.5 %)
Coefficient of Variation, $\bar{\sigma}_e / \overline{E(e_1 k_1)}$	0.87
Pre-tax 30-year risk-free real return on mean path, r_1 (Corresponding annual return)	143 % (3 %)
Rate of 30-year labor-augmenting tech. progress on mean path (Corresponding annual return)	143 % (3 %)
Debt-capital ratio, \bar{d}	25 %
Tax rate on capital income on constant growth path, τ_1^K	20 %
Social Security pay-as-you-go liabilities tax rate, $\tau_{s \geq 0}^{SS,P}$	11.5 %
Social Security funded portion tax rate, $\tau_{s \geq 0}^{SS,F}$	0.5 %
Implied Endogenous Variables (same in all simulations, unless indicated otherwise)	
Net national saving rate	4.4 %
“On Budget” Spending as a fraction of GDP on mean path, $G_0 / [A_0 k_0^\alpha]$	15.3 %
Capital income tax revenue as a fraction of GDP on mean path	4.8 %
Non-Social Security wage income tax revenue as a fraction of GDP on mean path	10.5 %
Exogenous Parameter (only for the benchmark)	
Correlation between capital income returns and wages on mean path	0.75

Table 2
Eliminating / Doubling Capital Income Tax Rates:

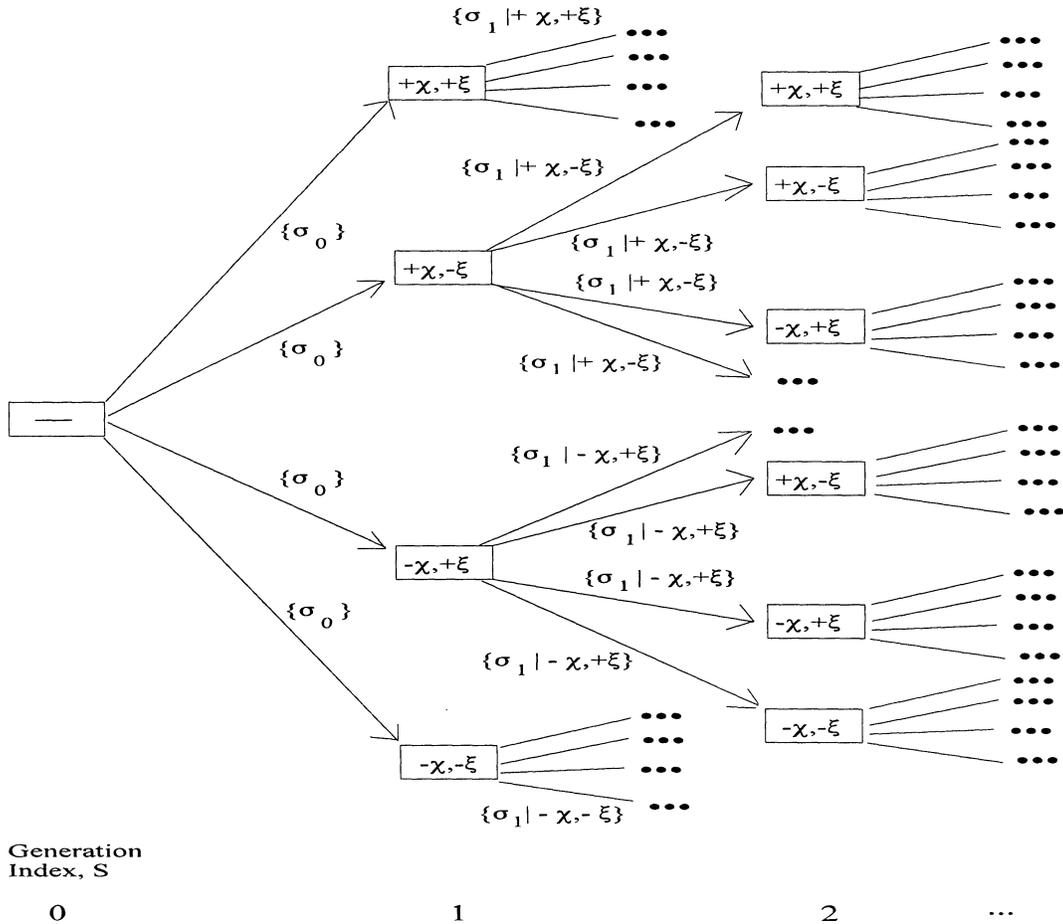
Changes in Macroeconomic Variables Along the Mean Growth Path^{1,2}

Generation Index ³	Percent Changes				Levels (in percent)			
	Capital Stock	Pre-tax Wages	Post-tax Wages ⁴	National Income	Wage Tax Rates	Risk-Free Rate (Annual)	Expected Return to Equities (Annual)	Equity Premium (Annual) ⁵
Decrease the Capital Income Tax from 20 to 0 Percent								
0	0.0	0.0	-0.3	0.0	15.2	3.0	8.8	5.8
1	-9.8	-3.1	-11.6	-2.9	21.4	2.0	8.8	6.8
2	-10.9	-3.4	-12.3	-3.2	21.8	2.7	8.8	6.1
3	-11.0	-3.4	-12.4	-3.3	21.8	2.8	8.8	6.0
4	-11.0	-3.4	-12.5	-3.3	21.8	2.8	8.8	6.0
5	-11.0	-3.5	-12.5	-3.3	21.8	2.8	8.8	6.0
Increase the Capital Income Tax from 20 to 40 Percent								
0	0.0	0.0	0.6	0.0	14.6	3.0	8.1	5.1
1	19.6	5.5	5.5	5.2	15.0	5.4	8.0	2.6
2	23.1	6.4	7.8	6.1	14.1	5.1	8.0	2.9
3	24.5	6.8	8.8	6.4	13.6	4.9	7.9	3.0
4	25.1	6.9	9.3	6.6	13.4	4.8	7.9	3.1
5	25.4	7.0	9.5	6.6	13.3	4.8	7.9	3.1

Notes:

1. I.e., state variables are updated between generations conditional on all shocks (both productivity and depreciation) taking their mean values *ex post*.
2. Calculations correspond to the benchmark model and calibration shown in Table 1 discussed in the Appendix.
3. Recall that each generation represents 30 years. Generation 0 is the initial young at the time of the policy change. They are allowed to re-optimize their portfolio and saving decisions in response to a policy change, including announcing a change in the capital income tax rate to be applied at time 1 during their second period of life. Generation -1 agents represent the elderly at the time of the reform and their saving and portfolio decisions and after-tax asset returns have already been determined by the time of the policy change.
4. I.e., after federal and Social Security taxes.
5. The equity premium equals 5.5 percent (annual) along the pre-reform constant growth path, reflecting a pre-reform expected return to equities of 8.5 percent (annual). The equity premium faced by generation-0 agents changes as the asset return distributions change in response to their re-optimization.

Figure 2
 Lattice Representation of Two-State Discrete Markov
 Chains for Productivity and Depreciation ^{1,2}



Notes:

1. σ_s is the vector of state variables at generation s .
2. Recall that (detrended) productivity can take the shock values $\{+\chi, -\chi\}$ and depreciation can take the shock values $\{+\xi, -\xi\}$.

Table 3
Eliminating / Doubling Capital Income Tax Rates:
—
Risk-Adjusted Changes in Expected Lifetime Resources of Generation¹

Generation Index, s	Percent Welfare Gain ² $[(\mu_{s t=0} - 1) \cdot 100\%]$
Decrease the Capital Income Tax from 20 to 0 Percent	
0	0.2
1	- 8.5
2	- 8.6
3	- 8.4
4	- 8.4
5	- 8.4
Increase the Capital Income Tax from 20 to 40 Percent	
0	3.5
1	- 1.6
2	- 1.3
3	- 1.5
4	- 1.9
5	- 2.1

Notes:

1. Calculations correspond to the benchmark model and calibration shown in Table 1 discussed in the Appendix.
2. I.e., generation s is indifferent between the policy change and a $[(\mu_{s|t=0} - 1) \cdot 100\%]$ percent increase in each possible wage at time s , measured today at time 0. Welfare measures are calculated over all possible paths that the economy can take between the policy reform date (0) and the shown "Generation Index." See the text for more details.

(Table 3 Continued on Next Page)

Table 3 Cont.

Generation Index, s	Percent Welfare Gain [[$\mu_{s t=0} - 1$) · 100%]
Increase the Capital Income Tax from 20 to 30 Percent	
0	1.2
1	1.5
2	2.7
3	2.6
4	2.5
5	2.4

Table 4
Correlation Between Wage and Stock Returns Equal to Unity ¹
 —
Risk-Adjusted Changes in Expected Lifetime Resources of Generation

Generation Index, s	Percent Welfare Gain $[(\mu_{s t=0} - 1) \cdot 100\%]$
Decrease the Capital Income Tax from 20 to 0 Percent	
0	0.0
1	0.4
2	0.7
3	0.9
4	1.0
5	1.1
Decrease the Capital Income Tax from 20 to -20 Percent:	
0	- 0.1
1	0.6
2	1.0
3	1.2
4	1.4
5	1.5

Notes:

1. The model is recalibrated to hit this correlation target and the other targets mentioned in the text.